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ON THE MOTION OF PISTON IN A POLYTROPIC GAS

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The motion of one-dimensional piston in a quiescent polytropic gas is considered. Approximate definition is obtained for the shock wave and the flow of gas behind the wave. This paper is related to [1].

1. We assume that a quiescent polytropic gas of density $\rho = \rho_+(x)$ and entropy $S = S_+(x)$ whose equation of state is $P = \rho^\gamma S$ lies to the right of a flat piston, and that $\rho_+(x)$ and $S_+(x)$ are analytic functions.

At instant of time $t = 0$ the piston begins to move in conformity with the law

$$x(t) = \xi_1 t + \xi_2 t^2 + \ldots + \xi_n t^n, \quad \xi_1 > 0$$

A shock wave propagates through the gas.

We seek the shock wave definition of the form

$$x = c_1 t + c_2 t^2 + \ldots + c_n t^n \tag{1.1}$$

The problem reduces to the determination of $c_1, \ldots, c_n$ in conformity with the given law of piston motion and the flow field between the shock wave and the piston. We shall show the feasibility of consecutive unique determination of $c_1, \ldots, c_n$ for any n. For the definition of flow between the piston and the shock wave for small t we use the series

$$u = \sum_{k=0}^{\infty} u_k(t)\varphi^k(x, t), \quad \rho = \sum_{k=0}^{\infty} \rho_k(t)\varphi^k(x, t) \tag{1.2}$$

$$S = \sum_{k=0} S_k(t)\varphi^k(x, t) \quad (\varphi(x, t) = x - c_1 t - \ldots - c_n t^n)$$

These series differ from those in [1] by that $\varphi(x, t) = 0$ is not a characteristic and from Kowalewska's series [2] by that the function $\varphi(x, t)$ is not a priori known.

The procedure for the determination of coefficients $c_1, \ldots, c_n$ and of the flow field is as follows. First, we represent coefficients u_k, ρ_k, and S_k $(k = 0, \ldots, n-1)$ of series (1.2) as functions of c_k $(k = 1, \ldots, n)$ with the latter determined by the specified conditions at the piston. Having determined c_k $(k = 1, \ldots, n)$ we obtain u_k, ρ_k, and S_k $(k = 0, \ldots, n-1)$.

The efficacity of this procedure will be shown on an example.

The proposed formulas may be used for determining the gas flow initial stage with subsequent application of difference methods.

We pass to detailed exposition.

The flow of gas between the piston and the shock wave satisfies the system equations of gasdynamics

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0 \tag{1.3}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\gamma S\rho^{\gamma-2}\frac{\partial\rho}{\partial x}+\rho^{\gamma-1}\frac{\partial S}{\partial x}=0$$

$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x}=0$$

We seek the solution of system (1.3) in the form of series (1.2) in which u_0, ρ_0, and S_0 are determined by the Hugoniot conditions [3]

$$u_0=(1-h)C_0\left(M_0-\frac{1}{M_0}\right),\quad h=\frac{\gamma-1}{\gamma+1},\quad M_0=\frac{D}{C_0}$$

$$\rho_0=\rho_+M_0^2\,[(1-h)+hM_0^2]^{-1}$$

$$S_0=M_0^{-2\gamma}S_+\,[(1+h)M_0^2-h]\,[(1-h)+hM_0^2]^{\gamma}$$

where C_0 is the speed of sound in the quiescent gas and D is the shock wave velocity.

We substitute series (1.2) into system (1.3) and equate to zero the coefficients at powers of φ. Coefficients u_k, ρ_k, and S_k $(k=1,\ldots)$ are functions of $c_1,\ldots,c_n$ and are determined successively as solutions of the system of linear algebraic equations with the determinant

$$\Delta=\det\begin{Vmatrix}\varphi_t'+u_0 & \rho_0 & 0\\ \gamma\rho_0^{\gamma-2}S_0 & \varphi_t'+u_0 & \rho_0^{\gamma-1}\\ 0 & 0 & \varphi_t'+u_0\end{Vmatrix}$$

Direct calculations show that when $c_k\neq\infty$ $(k=2,\ldots,n)$ the determinant Δ is positive in the neighborhood of point $\{t=0,\ x=0\}$, since c_1 determined by the condition $u_0(0)=\xi_1$ is of the form

$$c_1=\frac{\xi_1}{2(1-h)}+\left[\frac{\xi_1^2}{4(1-h)^2}+C_0^2(0)\right]^{1/2}$$

The condition $u(x(t),t)=\partial x/\partial t$ ($x(t)$ is the law of motion of the piston) is satisfied at the piston.

Differentiating the last equality with respect to t, we obtain a system of equations

$$\left.\frac{d^{j-1}}{dt^{j-1}}u(x(t),t)\right|_{t=0}=j!\,\xi_j$$

that can be used for successive determination of all c_j.

Substituting series (1.2) for u into the the j-th equation, we obtain

$$\frac{d^{j-1}u_0}{dt^{j-1}}+\frac{d^{j-2}u_1}{dt^{j-2}}(\varphi(x(t),t)_t'+\ldots+(j-1)u_{j-1}[(\varphi(x(t),t))_t']^{j-1}+$$
$$\Psi(c_1,\ldots,c_{j-1})\,|_{t=0}=j!\,\xi_j$$

where $(\Psi(c_1,\ldots,c_{j-1})$ is a known function.

It can be shown that the coefficient at c_j in the expression $u_k(t)\,(\varphi_t'(0,0))^k$ is of the form $d_kt^{j-(k+1)}$, where d_k is some positive constant. The coefficient at c_j in the considered equation is consequently positive. All c_j are uniquely determined.

Let us now consider the case of $n = 2$ and $\varphi = x - c_1 t - c_2 t^2$. We have

$$u_1(0) = \left[-\frac{\partial u_0(0)}{\partial t}(u_0(0) - c_1) + \gamma S_0(0)\rho_0^{\gamma-1}(0)\frac{\partial \rho_0(0)}{\partial t} + \frac{\partial S_0(0)}{\partial t}\rho_0^{\gamma-1}(0)\right][(u_0(0) - c_1)^2 - \gamma S_0(0)\rho_0^{\gamma-1}(0)]^{-1} \tag{1.4}$$

The derivatives in the expression for $u_1(0)$ may be represented in the form

$$\frac{\partial u_0}{\partial t}(0) = a_1 + b_1 c_2, \qquad \frac{\partial \rho_0}{\partial t}(0) = a_2 + b_2 c_2$$

$$\frac{\partial S_0}{\partial t}(0) = a_3 + b_3 c_2$$

Expressions for constants a_i and b_i $(i = 1, 2, 3)$ are not shown owing to their unwieldiness. Note that when density ρ_+ is constant, parameters a_1, a_2, and a_3 vanish. Let us determine c_2. The condition at the piston implies

$$2\xi_2 = \Phi_1(0) + \Phi_2(0)c_2$$

$$\Phi_1(0) = f(a_1, a_2, a_3), \qquad \Phi_2(0) = f(b_1, b_2, b_3)$$

$$f(x, y, z) = x + [(c_1 - u_0(0))x + \gamma S_0(0)\rho_0^{\gamma-2}(0)y + \rho_0^{\gamma-1}(0)z](u_0(0) - c_1)[(u_0(0) - c_1)^2 - \gamma\rho_0^{\gamma-1}(0)S_0(0)]^{-1}$$

Since $\Phi_2(0) > 0$, c_2 can be determined by formula

$$c_2 = \frac{2\xi_2 - \Phi_1(0)}{\Phi_2(0)} \tag{1.5}$$

which makes it possible to find some conclusions about the properties of the flow of gas behind the shock wave.

When the gas density ρ_+ is constant, the shock wave is accelerated at the initial instant $t = 0$ only if $\xi_2 \neq 0$, and the direction of acceleration is determined by the sign of ξ_2. In the case of distributed density the shock wave is accelerated when $\xi_2 = 0$. The dependence of acceleration on initial data is complex. Since $\Phi_1(0)$ is independent of ξ_2, hence setting $\xi_2 = {}^1/_2\Phi_1(0)$ we obtain $c_2 = 0$. Although the acceleration of the piston at $t = 0$ is nonzero, that of the shock wave is zero.

Let us determine what happens to the shock wave when $\xi_1 \to 0$. By the Hugoniot condition $c_1(\xi_1) \to C_0(0)$ when $\xi_1 \to 0$. Let us determine the $\lim c_2(\xi_1)$ when $\xi_1 \to 0$ using formula (1.5). Calculations show that then

$$a_1 \to 2C_0(0)\frac{\partial C_0}{\partial x}(0)(1 - h), \qquad b_1 \to 2(1 - h)$$

$$C_0^2 \to \gamma\rho_+^{\gamma-1}S_+ = \gamma\rho_+^{-1}, \qquad \gamma S_0\rho_0^{\gamma-2}a_2 + a_3\rho_0^{\gamma-1} \to 0$$

$$S_0\rho_0^{\gamma-2}b_2 + b_3\rho_0^{\gamma-1} \to 0, \qquad (u_0 - c_1)^2 - \gamma\rho_0^{\gamma-1}S_0 \to 0$$

Hence

$$\lim_{\xi_1 \to 0} c_2(\xi_1) = \lim_{\xi_1 \to 0}\frac{-a_1}{b_1} = C_0(0)\frac{\partial C_0}{\partial x}(0)$$

Note that $C_0(0)$ is the velocity and $C_0(0)\,\partial C_0(0)/dx$ is the acceleration of a weak discontinuity at the initial instant. Thus in the considered approximation the shock wave degenerates into a weak discontinuity.

We shall show that in this case the flow of gas behind the shock wave may be defined by the solution of the problem with a weak discontinuity [1], i.e.

$$\frac{\partial u}{\partial x}(0,0) \to \frac{-2\xi_2}{C_0}, \qquad \frac{\partial u}{\partial t}(0,0) \to 2\xi_2, \quad \xi_1 \to 0$$

The conditions at the piston imply that

$$2\xi_2 = \frac{\partial u_0(0)}{\partial t} + u_1(0)\ (u_0(0) - c_1)$$

and from fromula (1,4) we have

$$\frac{\partial u(0,0)}{\partial x} = u_1(0) = \frac{2\xi_2 - \partial u_0(0) / \partial t}{u_0(0) - c_1} \to \frac{2\xi_2}{-C_0}, \quad \xi_1 \to 0$$

$$\frac{\partial u(0,0)}{\partial t} = \frac{\partial u_0(0)}{\partial t} - u_1(0)\, c_1 \to 2\xi_2, \quad \xi_1 \to 0$$

Example. Let the piston motion be defined by $x = 10t + 5t^2$ and $\rho_+ = S_+ = 1$, $u_+ = 0$, and $P = \rho^2 S$. We seek a solution of the shock wave of the form $x = c_1 t + c_2 t^2$. After necessary calculations we obtain the following law of shock wave motion

$$x = 15.132t + 4.241t^2 \tag{1.6}$$

Solution (1.6) was compared with that obtained for the same example by the difference method up to time $t = 0.5$. The discrepancies were found to be less than 0.1% for $t < 0.3$, and less than 1% for $0.3 < t < 0.5$.

2. So far only plane one-dimensional motion of the piston was considered. Let us now consider cylindrical and spherical motions of the piston in addition to its plane motion. The first of equations of system (1.3) is now replaced by

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\nu \rho u}{x} = 0$$

where $\nu = 0$ relates to plane motion, $\nu = 1$ to cyndrical and $\nu = 2$ to spherical motions.

Let the piston motion be defined by the formula

$$x = x_0 + \xi_1 t + \ldots + \xi_n t^n$$

where x_0 and ξ_i $(i = 1, \ldots, n)$ are constants, and $x_0 > 0$ and $\xi_1 > 0$ (in Sect. 1 $x_0 = 0$).

We seek the shock wave definition in the form $x = x_0 + c_1 t + \ldots + c_n t^n$, and that of the flow of gas between the piston and the shock wave in the form of series

$$\rho = \rho_0(t) + \sum_{k=1}^{\infty} \rho_k(x,t)\, \varphi^k(x,t) \tag{2.1}$$

$$u = u_0(t) + \sum_{k=1}^{\infty} u_k(x,t)\, \varphi^k(x,t)$$

$$S = S_0(t) + \sum_{k=1}^{\infty} S_k(x,t)\, \varphi^k(x,t)$$

$$(\varphi = x - x_0 - c_1 t - \ldots - c_n t^n)$$

where values of ρ, u, and S at the shock wave are denoted by subscript 0.

Series (2.1) differ from series (1.2) by that in the former coefficients ρ_k, u_k, and S_k $(k = 1, \ldots)$ depend not only on t, but also on x. Parameters $c_1, \ldots, c_n$, ρ_k, u_k, and S_k $(k = 1, \ldots)$ are determined in exactly the same way as in the plane case.

Example. Let us assume that the parameters of gas in which the piston is moving are: $\rho_+ = S_+ = 1$ and $P = \rho^2 S$ $(\nu = 2)$ and that the shock wave generated by it is strong. Then, after necessary calculations, for the coefficients c_1 and c_2 we obtain the following formulas:

$$c_1 = 1.5\ \xi_1, \quad c_2 = 0.75\,(\xi_2 - 0.75\nu\xi_1^2) \tag{2.2}$$

which shows that the propagation of the spherical wave $(\nu = 2)$ is the slowest.

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